

THE WEINSTEIN CONJECTURE FOR CONNECTED SUMS

HANSJÖRG GEIGES AND KAI ZEHMISCH

ABSTRACT. We prove the Weinstein conjecture for non-trivial contact connected sums under either of two topological conditions: non-trivial fundamental group or torsion-free homology.

1. INTRODUCTION

A $(2n + 1)$ -dimensional closed manifold M is said to satisfy the Weinstein conjecture if the Reeb vector field of any contact form on M has a periodic orbit. For a recent survey on the status of this conjecture see [10]. In his seminal paper [6] on the Weinstein conjecture in dimension three, Hofer proved the Weinstein conjecture for overtwisted 3-manifolds and for any closed, orientable 3-manifold with non-vanishing second homotopy group; in fact, in either of these cases there exists a *contractible* periodic Reeb orbit.

These results can be combined into the following statement.

Theorem 1 (Hofer). *Let $(M, \xi) = (M_1, \xi_1) \# (M_2, \xi_2)$ be the contact connected sum of two closed, connected contact manifolds of dimension 3. Suppose we can write $\xi = \ker \alpha$ with a contact form α whose Reeb vector field does not have any contractible periodic orbits. Then one of the summands (M_i, ξ_i) is contactomorphic to the 3-sphere S^3 with its standard tight contact structure ξ_{st} .*

Proof. Consider the belt sphere $S^2 \subset M$ of the connected sum. Since there are no contractible Reeb orbits, the contact structure ξ must be tight, and by a perturbation of S^2 one may achieve that the characteristic foliation of S^2 induced by ξ has two elliptic singular points connected by all the leaves of the foliation, cf. [3, Section 4.6]. Under the assumption of the non-existence of contractible periodic orbits, Hofer then produces a filling of S^2 by holomorphic discs in the half-symplectisation $(-\infty, 0] \times M$, which projects to a 3-cell in M bounded by S^2 . Together with the Poincaré conjecture and Eliashberg's classification of tight contact structures on S^3 , one concludes that one of the (M_i, ξ_i) equals (S^3, ξ_{st}) . \square

Our aim is to prove an analogous result in higher dimensions.

Theorem 2. *Let $(M, \xi) = (M_1, \xi_1) \# (M_2, \xi_2)$ be the contact connected sum of two closed, connected contact manifolds of dimension $2n + 1 \geq 5$. Suppose we can write $\xi = \ker \alpha$ with a contact form α whose Reeb vector field does not have any contractible periodic orbits. Assume further that one of the following conditions is satisfied:*

- (i) *M is simply connected and has torsion-free homology, or*

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(ii) M is not simply connected.

Then one of the summands M_1, M_2 is homeomorphic to S^{2n+1} .

Put another way, if (M, ξ) is a non-trivial connected sum (i.e. neither summand is a homotopy sphere) and M satisfies condition (i) or (ii), then any contact form defining ξ will have a contractible periodic Reeb orbit. This result has the potential to serve as a contact-geometric primality test for manifolds. For instance, if (Q, g) is a Riemannian manifold without contractible periodic geodesics (say, a manifold with non-positive sectional curvature), then the unit cotangent bundle ST^*Q with its canonical contact form does not have any contractible periodic Reeb orbits, since the Reeb flow equals the cogeodesic flow, cf. [3, Section 1.5]. If ST^*Q satisfies one of the topological assumptions (i) or (ii), it cannot be a non-trivial contact connected sum.

In their work on the planar circular restricted three-body problem, Albers et al. [1] show that the energy hypersurface for energies slightly above the energy of the first Lagrange point is the contact connected sum of two copies of \mathbb{RP}^3 with its standard tight contact structure. Hofer's Theorem 1 then implies the existence of a periodic orbit. Our Theorem 2 may have similar applications.

The germ of the contact structure $\xi = \ker \alpha$ near a compact hypersurface $S \subset M$ is determined by the 1-form $\alpha|_{TS}$, cf. [2, Proposition 6.4]. In a contact connected sum as in [12], cf. [3, Chapter 6], the contact form along the belt sphere S^{2n} equals $\frac{1}{2}(\mathbf{x} d\mathbf{y} - \mathbf{y} d\mathbf{x})|_{TS^{2n}}$, where S^{2n} is regarded as a subset of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ with coordinates $(w, \mathbf{x}, \mathbf{y})$. We shall call a $2n$ -sphere with a germ of a contact structure equal to this one a **standard** $2n$ -sphere. Conversely, given a *separating* standard $2n$ -sphere in a contact manifold, the manifold is a contact connected sum.

The following theorem tells us that under the assumption on the non-existence of contractible periodic Reeb orbits, the property of S^{2n} being separating follows automatically from a knowledge of the contact structure near S^{2n} . So in Theorem 2 it would suffice to require the existence of any standard $2n$ -sphere.

Theorem 3. *Let $(M, \xi = \ker \alpha)$ be a closed $(2n+1)$ -dimensional contact manifold, $n \geq 2$, such that α does not have any contractible periodic Reeb orbits. If $S^{2n} \subset M$ is a standard sphere, then S^{2n} separates, so that (M, ξ) is a contact connected sum.*

This immediately leads to the following corollary.

Corollary 4. *If a contact manifold (M, ξ) is obtained by an index 1 surgery on a closed, connected contact manifold (M', ξ') , then any contact form defining ξ must have contractible periodic Reeb orbits.*

Proof. The belt sphere of the 1-handle corresponding to the surgery does not separate M . Now apply Theorem 3. \square

For example, the standard contact structure on $S^1 \times S^{2n}$ is obtained by an index 1 surgery on the standard S^{2n+1} . So any contact form defining this contact structure must have a contractible periodic Reeb orbit.

The strategy for proving Theorem 3 is to show that the moduli space of suitable holomorphic discs, which will be compact under the assumption on the non-existence of contractible periodic Reeb orbits, leads to a filling of S^{2n} inside M , i.e. S^{2n} is homologically trivial. With the additional assumption (i) or (ii) in Theorem 2, this filling can be shown to be a ball.

We start with a moduli space \mathcal{W} of holomorphic discs with Lagrangian boundary condition inside a half-symplectisation W of (M, ξ) . By deforming the evaluation map $\text{ev}: \mathcal{W} \times \mathbb{D} \rightarrow W$ to a map into M which sends the boundary $\partial(\mathcal{W} \times \mathbb{D})$ onto the belt sphere of the index 1 surgery, we arrive at the desired topological conclusions.

A different approach to the type of questions addressed by Theorems 2 and 3 has been suggested by Niederkrüger [9] in collaboration with Ghiggini and Wendl.

2. CONTACT SURGERY OF INDEX 1

2.1. The model handle. Given a standard sphere $S^{2n} \subset (M, \xi)$ as assumed in Theorem 3, a neighbourhood of it can be identified with a neighbourhood of the belt sphere in the standard picture for an index 1 surgery, see for instance [3, Figure 6.4]. This allows one to perform a reverse contact surgery on that sphere. In other words, (M, ξ) is the result of performing an index 1 surgery on a suitable contact manifold (M', ξ') .

The standard contact structure $\ker(dw + \frac{1}{2}(\mathbf{x} d\mathbf{y} - \mathbf{y} d\mathbf{x}))$ on \mathbb{R}^{2n+1} is preserved by the map $(w, \mathbf{x}, \mathbf{y}) \mapsto (\lambda^2 w, \lambda \mathbf{x}, \lambda \mathbf{y})$; the homothety $(w, \mathbf{x}, \mathbf{y}) \mapsto (w, \mathbf{x}, \mathbf{y})/R$ pulls back the standard contact form to $(R dw + \frac{1}{2}(\mathbf{x} d\mathbf{y} - \mathbf{y} d\mathbf{x}))/R^2$. Therefore we may assume that the index 1 surgery on (M', ξ') is performed inside two balls of large radius R , with contact form $\pm R dw + \frac{1}{2}(\mathbf{x} d\mathbf{y} - \mathbf{y} d\mathbf{x})$ for ξ' on the respective ball.

Here is an explicit description of this surgery. We slightly modify the picture in [12] and [3] so as to obtain a good model for the holomorphic analysis.

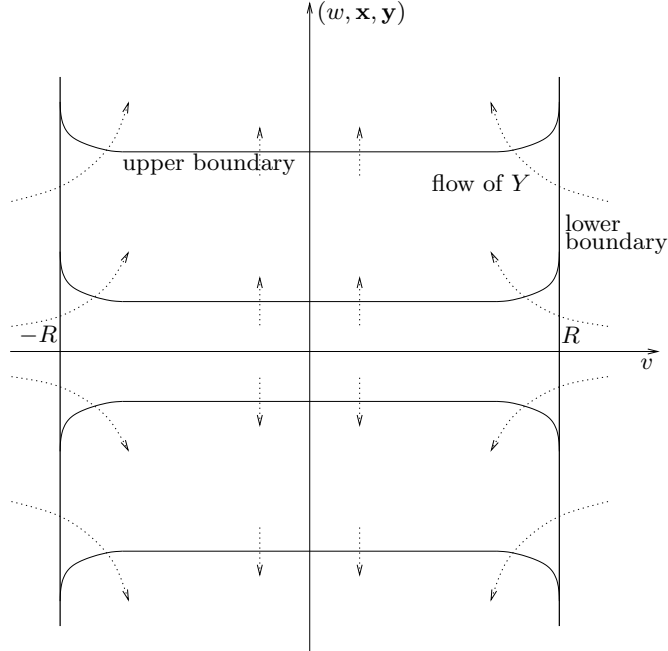


FIGURE 1. The surgery model

On \mathbb{R}^{2n+2} with coordinates $v, w \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we consider the symplectic form $\omega_0 = dv \wedge dw + d\mathbf{x} \wedge d\mathbf{y}$ and the Liouville vector field

$$Y_0 = -v \partial_v + 2w \partial_w + \frac{1}{2}(\mathbf{x} \partial_{\mathbf{x}} + \mathbf{y} \partial_{\mathbf{y}}).$$

This vector field is transverse to the hypersurfaces $\{v = \pm R\}$, where it induces the contact form

$$i_{Y_0} \omega_0 = \mp R dw + \frac{1}{2}(\mathbf{x} d\mathbf{y} - \mathbf{y} d\mathbf{x}),$$

so we may identify balls of Radius R in $\{v = \pm R\}$ with the two Darboux charts in (M', ξ') where we want to perform the surgery. This surgery is performed by replacing two balls of radius $r < R$ in these charts with a cylinder

$$\{(v, w, \mathbf{x}, \mathbf{y}) : w^2 + \frac{1}{2}(|\mathbf{x}|^2 + |\mathbf{y}|^2) = r^2\}$$

and rounding corners, see Figure 1. That figure shows a ‘thick’ handle with r close to R , and a ‘thin’ handle with $r = 1$, say. Since Y_0 is everywhere transverse to the new hypersurface, it induces a contact structure on the manifold obtained by index 1 surgery. On the ‘straight’ part of the cylinder we glued in, i.e. away from a small region where we had to round corners, we would like to replace Y_0 by the ‘radial’ Liouville vector field

$$Y_1 = w \partial_w + \frac{1}{2}(\mathbf{x} \partial_{\mathbf{x}} + \mathbf{y} \partial_{\mathbf{y}}).$$

This Y_1 has the advantage of being the gradient vector field of the strictly plurisubharmonic function

$$(v, w, \mathbf{x}, \mathbf{y}) \mapsto \frac{1}{2}w^2 + \frac{1}{4}(|\mathbf{x}|^2 + |\mathbf{y}|^2)$$

on \mathbb{C}^{n+1} with coordinates $v + iw, \mathbf{x} + i\mathbf{y}$, so the induced contact structure on the cylinder is given by the tangent hyperplanes invariant under the standard complex structure i on \mathbb{C}^{n+1} .

It is indeed possible to interpolate between the Liouville vector fields Y_0 near $v = \pm R$ and Y_1 for $|v| < R - \varepsilon$ via Liouville vector fields transverse to the boundary of the handle. Let $v \mapsto a(v)$ be a smooth function interpolating monotonically between $a \equiv -1$ for $|v| \leq R - \varepsilon$ and $a \equiv 0$ for $|v| \geq R - \varepsilon/2$. Then modify Y_0 by adding the Hamiltonian vector field X_H of the function $H(v, w, \mathbf{x}, \mathbf{y}) = a(v) \cdot vw$. This will certainly produce a Liouville vector field Y , and it remains to check the other desired properties.

We have

$$X_H = -\frac{\partial H}{\partial w} \partial_v + \frac{\partial H}{\partial v} \partial_w = -a(v) \cdot v \partial_v + (a'(v) \cdot vw + a(v) \cdot w) \partial_w,$$

hence

$$\begin{aligned} Y &= Y_0 + X_H \\ &= -(1 + a(v)) \cdot v \partial_v + (2 + a'(v) \cdot v + a(v)) \cdot w \partial_w + \frac{1}{2}(\mathbf{x} \partial_{\mathbf{x}} + \mathbf{y} \partial_{\mathbf{y}}). \end{aligned}$$

So Y interpolates between Y_0 for $|v| \geq R - \varepsilon/2$ and Y_1 for $|v| \leq R - \varepsilon$ as desired. Moreover, the coefficient of $w \partial_w$ is always positive, which guarantees the transversality requirement.

A similar construction can be found in [3, Theorem 7.6.2].

2.2. The symplectic cobordism. Write M_1, M_R for the copies of M obtained by attaching the thin resp. thick handle to M' . We write α_1 for a contact form on M_1 inducing the contact structure ξ' away from the thin handle and equal to $i_Y\omega_0$ on the upper boundary of the thin handle, so that $(M_1, \ker \alpha_1)$ is contactomorphic to (M, ξ) . Similarly, we define α_R on M_R . Let α be any other contact form on M with $\ker \alpha = \xi$.

We now build a symplectic manifold (W, ω) as follows. Write $\alpha_1 = e^g \alpha$ with a suitable smooth function $g: M \rightarrow \mathbb{R}$. Since M is compact, we can choose an $a_0 \in \mathbb{R}^+$ such that $e^{-a_0} < e^g$. Then

$$\{(a, x) \in \mathbb{R} \times M : -a_0 \leq a \leq g(x)\}$$

with symplectic form $d(e^a \alpha)$ and Liouville vector field ∂_a defines a cylindrical Liouville cobordism between

$$(\{-a_0\} \times M, e^{-a_0} \alpha) \text{ and } (M_1, \alpha_1).$$

To the concave end $\{-a_0\} \times M$ of this symplectic cobordism, we can attach the half-symplectisation

$$((-\infty, -a_0] \times M, d(e^a \alpha)).$$

Finally, to the convex end M_1 of the cylindrical Liouville cobordism we attach the part of the handle between the thin and the thick upper boundary.

The resulting symplectic manifold (W, ω) admits a globally defined and nowhere vanishing Liouville vector field inducing the appropriate contact form on the transversals $\{-a_0\} \times M$, M_1 , and M_R .

2.3. The almost complex structure. We choose an ω -compatible almost complex structure J on W with the following properties:

- (J1) On $(-\infty, -a_0] \times M$ we choose an \mathbb{R} -invariant almost complex structure J preserving $\ker \alpha$ and sending ∂_a to the Reeb vector field of α .
- (J2) On the part between the thin and the thick handle, i.e. between M_1 and M_R , we choose $J = i$ for $|v| \leq R - \varepsilon$, and generic elsewhere, subject to the condition that M_R be a (strictly) J -convex boundary.
- (J3) On $[-a_0, 0] \times M$ we choose any generic almost complex structure extending the choices in (J1) and (J2).

3. THE MODULI SPACE OF HOLOMORPHIC DISCS

3.1. The Lagrangian boundary condition. We are now going to describe a family of Lagrangian cylinders in \mathbb{C}^{n+1} . Their intersection with the part of (W, ω) between M_1 and M_R — which for ease of reference we shall simply refer to as the ‘handle’ — will provide the boundary condition for the holomorphic discs we wish to study.

Recall that on \mathbb{C}^{n+1} we have the complex coordinates $v + iw, \mathbf{x} + i\mathbf{y}$; we now write the latter as $\mathbf{x} + i\mathbf{y} = \mathbf{h} + z_n$ with $\mathbf{h} \in \mathbb{C}^{n-1}$ and $z_n \in \mathbb{C}$. For each $\mathbf{t} \in \mathbb{R}^{n-1}$, the Lagrangian cylinder

$$L^{\mathbf{t}} := (\{0\} \times \mathbb{R}_w) \times (\mathbb{R}_{\text{Re}(\mathbf{h})}^{n-1} \times \{\mathbf{it}\}) \times \partial D_{z_n}^2 \subset \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}$$

is filled by standard holomorphic discs

$$\mathbb{D} \ni z \mapsto u_{\mathbf{s}, w}^{\mathbf{t}}(z) := (0, w; \mathbf{s} + i\mathbf{t}; z),$$

where \mathbb{D} denotes the unit disc in \mathbb{C} , and the parameters w, \mathbf{s} range over \mathbb{R} and \mathbb{R}^{n-1} , respectively.

3.2. Definition of the moduli space. We define \mathcal{W} to be the moduli space of holomorphic discs $u: \mathbb{D} \rightarrow W$ (smooth up to the boundary) with $u(\partial\mathbb{D}) \subset L^{\mathbf{t}} \cap W$, where \mathbf{t} is allowed to vary over \mathbb{R}^{n-1} , and satisfying the following conditions:

- (M1) The relative homology class $[u] \in H_2(W, L^{\mathbf{t}} \cap W)$ equals that of some standard disc $u_{\mathbf{s}, w}^{\mathbf{t}}$ in W .
- (M2) The parametrisation of u is fixed by the requirement $u(i^k) \in L^{\mathbf{t}} \cap \{z_n = i^k\}$ for $k = 0, 1, 2$.

Observe that standard discs of the same level \mathbf{t} are homotopic in $(W, L^{\mathbf{t}} \cap W)$.

3.3. Properties of the holomorphic discs. The present set-up is similar to that in our previous paper [4], and the basic properties of the holomorphic discs are established by the same arguments as in [4, Section 3.3].

Proposition 5. *The discs $u \in \mathcal{W}$ have the following properties:*

- (i) *The Maslov index equals 2.*
- (ii) *The symplectic energy equals π .*
- (iii) *All discs are simple.*

3.4. Compactness. The bubbling of spheres is impossible in an exact symplectic manifold. Both bubbling of finite energy planes and breaking are prevented by the existence of a global non-singular Liouville vector field (and the resulting maximum principle) and the assumption that there be no contractible periodic Reeb orbits. The bubbling of discs at boundary points is ruled out by the Lagrangian boundary condition (which fixes the energy) and the homotopical condition (M2). This guarantees the compactness of \mathcal{W} . For more details see [4, Section 4].

3.5. Transversality. If a holomorphic disc $u \in \mathcal{W}$ has its image $u(\mathbb{D})$ contained entirely in the part of the handle determined by $|v| \leq R - \varepsilon$, where $J = i$, then u can be written globally in terms of holomorphic component functions $(v + iw, \mathbf{h}, z_n)$. The boundary conditions $v = 0$, $\text{Im}(\mathbf{h}) = \mathbf{t}$, $|z_n| = 1$, and $z_n(i^k) = i^k$ for $k = 0, 1, 2$ force $u = u_{\mathbf{s}, w}^{\mathbf{t}}$ for suitable \mathbf{s}, w . This follows from a simple application of the maximum and argument principles, cf. the proof of Lemma 9 in [4]. So for discs in this region, transversality is obvious.

Holomorphic discs that leave this region have to pass through a domain where J may be chosen generic, which ensures regularity for these discs, too.

No holomorphic disc can touch the J -convex boundary of W at an interior point, so the boundary of the moduli space is made up of discs u with $u(\partial\mathbb{D}) \cap \partial W \neq \emptyset$. We want to show that any such disc has to be standard. One may then conclude that \mathcal{W} is a manifold (with boundary) of the expected dimension $2n - 1$, determined by the Maslov index.

The following slightly stronger statement will be relevant for computing the degree of an evaluation map on the moduli space later on. It is understood that R has been chosen large.

Lemma 6. *Any non-standard disc in \mathcal{W} satisfies $w^2 + |\mathbf{h}|^2 \leq 8$ on $\partial\mathbb{D}$.*

Proof. Consider the open subset $G \subset \mathbb{D}$ of points mapping into that part of the handle where $|v| < R - \varepsilon$; this set G contains $\partial\mathbb{D}$. On the closure \overline{G} we consider

the holomorphic components $g := (v + iw, \mathbf{h})$ of u ; we suppress the component z_n . Notice that for a standard disc these components are constant.

If $\overline{G} = \mathbb{D}$, then u is standard by the argument preceding this lemma.

Let us now assume that \overline{G} is not all of \mathbb{D} . On the topological boundary ∂G of G in \mathbb{D} we have $|v| = R - \varepsilon$ or $w^2 + |\mathbf{h}|^2/2 + |z_n|^2/2 = 1$; in the latter case this means $w^2 + |\mathbf{h}|^2 \leq 2$. On $\partial\mathbb{D}$ we have $v = 0$ and $\text{Im}(\mathbf{h}) = \mathbf{t}$. This allows us to extend the holomorphic map g by Schwarz reflection to the compact subset S of the Riemann sphere given as the union of \overline{G} and its reflected copy, cf. the proof of [4, Lemma 11].

Choose a point s_0 in $\partial\mathbb{D}$ with $|(w, \mathbf{h})(s_0)|^2 = (\sqrt{2} + r_0)^2$. We want to show that $r_0 \leq \sqrt{2}$. The closed $2n$ -ball \overline{B}_r of radius r about the point $g(s_0)$ does not intersect $g(\partial S)$ for $r < r_0$. The monotonicity lemma [7, Theorem II.1.3] then says that the area of $g(S) \cap \overline{B}_r$ is bounded from below by πr^2 . So the area of $g(\overline{G}) \cap \overline{B}_r$ is bounded from below by $\pi r^2/2$, and above by the total energy π of u . It follows that $r \leq \sqrt{2}$. As a consequence, the same inequality must hold for r_0 . \square

Remark 7. With similar estimates one can show that in the part of the handle where $|v| < R/2$, say, a neighbourhood of the upper boundary of the handle contains standard discs only.

An orientation of \mathcal{W} is determined as in [4, Section 5.3]. This will allow us to speak of the degree of the evaluation map on \mathcal{W} .

4. PROOF OF THEOREM 3

Given a contact manifold $(M, \xi = \ker \alpha)$ satisfying the assumptions of Theorem 3, and a standard sphere $S^{2n} \subset M$, we can build a symplectic cobordism as in Section 2.2. Write $S_b \subset M_R$ for the belt sphere given as the intersection of the upper boundary of the thick handle with the hyperplane $\{v = 0\}$ in \mathbb{C}^{n+1} . This belt sphere can be described explicitly in the form

$$S_b = \{(v, w, \mathbf{h}, z_n) : v = 0, w^2 + \frac{1}{2}(|\mathbf{h}|^2 + |z_n|^2) = (R - \delta)^2\},$$

where $R - \delta$ was the level near R used to define the thick handle in a neighbourhood of $v = 0$. The pair (M, S^{2n}) is diffeomorphic to (M_R, S_b) , so it suffices to show that S_b is homologically trivial in M_R .

We are going to show that the composition $f_0 := \text{pr}_Y \circ \text{ev} : \mathcal{W} \times \mathbb{D} \rightarrow M_R$ of the evaluation map

$$\text{ev} : \mathcal{W} \times \mathbb{D} \longrightarrow W, \quad (u, z) \longmapsto u(z)$$

and the projection pr_Y of W onto M_R along the flow lines of Liouville vector field Y sends the boundary $\partial(\mathcal{W} \times \mathbb{D})$ with degree 1 onto S_b . This implies the homological statement we want to prove, and hence Theorem 3.

The boundary of $\mathcal{W} \times \mathbb{D}$ is made up of the parts $\partial\mathcal{W} \times \mathbb{D}$ and $\mathcal{W} \times \partial\mathbb{D}$. The boundary $\partial\mathcal{W}$ of the moduli space \mathcal{W} consists of standard discs $u_{\mathbf{s}, w}^{\mathbf{t}}$ whose boundary $u_{\mathbf{s}, w}^{\mathbf{t}}(\partial\mathbb{D})$ lies in S_b , i.e. where $w^2 + (|\mathbf{h}|^2 + 1)/2 = (R - \delta)^2$. The images $u_{\mathbf{s}, w}^{\mathbf{t}}(\mathbb{D})$ lie in the hyperplane $\{v = 0\}$. So when we project these images along the flow lines of the Liouville vector field Y to S_b , they foliate a neighbourhood $S^{2n-2} \times D^2 \subset S_b$ of

$$S^{2n-2} \times \{0\} = S_b \cap \{z_n = 0\}.$$

The part $\mathcal{W} \times \partial\mathbb{D}$ of the boundary is mapped by the evaluation map to points on some $L^{\mathbf{t}}$, where $v = 0$ and $|z_n| = 1$. Since Y is radial in (w, \mathbf{h}, z_n) -direction near

$v = 0$, the points in $f_0(\mathcal{W} \times \partial\mathbb{D})$ satisfy $|z_n| \geq 1$. In particular, this set is disjoint from a neighbourhood of $S_b^{2n-2} \subset S_b$.

It follows, as claimed, that the restriction

$$f_0|_{\partial(\mathcal{W} \times \mathbb{D})}: \partial(\mathcal{W} \times \mathbb{D}) \longrightarrow S_b$$

is of degree 1.

5. PROOF OF THEOREM 2

5.1. Deforming the evaluation map. The first step in the proof of Theorem 2, as in the proof of Theorem 3, is to construct a continuous map $\mathcal{W} \times \mathbb{D} \rightarrow M_R$ sending $\partial(\mathcal{W} \times \mathbb{D})$ with degree 1 onto S_b . This time, however, we should like to have more control over the behaviour of the map near the boundary $\partial(\mathcal{W} \times \mathbb{D})$. This greatly simplifies the subsequent topological arguments.

(1) Instead of projecting the whole image $\text{ev}(\mathcal{W} \times \mathbb{D})$ along the flow lines of Y to M_R , we first consider only the image of the boundary $\partial(\mathcal{W} \times \mathbb{D})$ under ev . For ease of notation, we write $\partial_+ := \text{ev}(\partial\mathcal{W} \times \mathbb{D})$ and $\partial_- := \text{ev}(\mathcal{W} \times \partial\mathbb{D})$, so that the image in question is given as the union $\partial_+ \cup \partial_-$.

This image lies in the intersection of the handle with the hyperplane $\{v = 0\}$. On that intersection, we can define a map, homotopic to the identity, which sends each segment of a flow line of Y starting in a point of ∂_+ and ending in a point of S_b to its endpoint. Notice that under this requirement, points in the set $\partial_+ \cap \partial_-$, which by Lemma 6 equals $\text{ev}(\partial\mathcal{W} \times \partial\mathbb{D})$, do not move. So we may require that the map do not move any points in ∂_- , and no points outside a small neighbourhood of ∂_+ inside the full image of $\mathcal{W} \times \mathbb{D}$ under ev . Using v as a homotopy parameter, we can extend this map to W , equal to the identity outside a small neighbourhood of $\{v = 0\}$. In this way we can ensure by Remark 7 that points in the image of any non-standard discs do not move. By postcomposing ev with this map, we obtain a map $f_1: \mathcal{W} \times \mathbb{D} \rightarrow W$ whose effect on ∂_+ is the same as that of the map f_0 considered in the proof of Theorem 3.

(2) Next we want to find a continuous map $f_2: \mathcal{W} \times \mathbb{D} \rightarrow W$ that sends $\partial(\mathcal{W} \times \mathbb{D})$ with degree 1 onto S_b (for the appropriate choice of orientations). In contrast with the map f_0 considered previously, we do not yet want any points in the interior of $\mathcal{W} \times \mathbb{D}$ to map to S_b (or in fact M_R). Since \mathcal{W} is compact, we can choose a small annular neighbourhood $A = [a, 1] \times S^1$ of $\partial\mathbb{D}$ in \mathbb{D} such that $f_1(\mathcal{W} \times A)$, when projected to M_R along the flow lines of Y , is contained in a prescribed small neighbourhood of S_b . For $r \in [a, 1]$ and $\varphi \in S^1$ write $\gamma_{u,r,\varphi}$ for the flow line of Y through the point $f_1(u, r, \varphi)$, reparametrised linearly such that $\gamma_{u,r,\varphi}(a) = f_1(u, r, \varphi)$ and $\gamma_{u,r,\varphi}(1)$ is the point on this flow line that lies in M_R . Now define $f_2: \mathcal{W} \times \mathbb{D} \rightarrow W$ by the requirements

$$f_2 = f_1 \text{ on } \mathcal{W} \times (\mathbb{D} \setminus \text{Int}(A))$$

and

$$f_2(u, r, \varphi) = \gamma_{u,r,\varphi}(r) \text{ for } (u, r, \varphi) \in \mathcal{W} \times [a, 1] \times S^1.$$

This map has the same effect on $\partial(\mathcal{W} \times \mathbb{D})$ as f_0 .

(3) The deformations we have considered thus far keep points in $\{v = 0\}$ inside that hyperplane. Hence, if u is a standard disc, then $f_2(u, z)$ will be a point in $W \cap \{v = 0\}$ for any $z \in \mathbb{D}$, and a point in $M_R \cap \{v = 0\}$ if and only if $u \in \partial\mathcal{W}$ or $z \in \partial\mathbb{D}$.

If u is a non-standard disc, then $u(A)$ will be far away from M_R by Remark 7. As a consequence, $f_2(u \times A)$ will meet M_R at a steep angle, i.e. the tangent vectors have a small v -component compared to a large (w, \mathbf{h}, z_n) -component.

Hence, by a homotopy of W that keeps M_R fixed and moves the interior of the handle near $\{v = 0\}$ in the direction of negative v , we obtain a new map $f_3: \mathcal{W} \times \mathbb{D} \rightarrow W$ with the property that all points in $f_3(\text{Int}(\mathcal{W}) \times \text{Int}(A))$ have negative v -component. On $\partial(\mathcal{W} \times \mathbb{D})$, the map has not been changed.

(4) Now postcompose f_3 with the projection pr_Y of W onto M_R along the flow lines of Y to obtain a map f_4 from $\mathcal{W} \times \mathbb{D}$ to M_R . Again, this has no further effect on $\partial(\mathcal{W} \times \mathbb{D})$, but we are now guaranteed that a neighbourhood of that boundary inside $\mathcal{W} \times \mathbb{D}$ is mapped by f_4 entirely to one side of the belt sphere S_b .

(5) Finally, define

$$f: (\mathcal{W} \times \mathbb{D}, \partial(\mathcal{W} \times \mathbb{D})) \longrightarrow (M_R, S_b)$$

as a smooth map homotopic to f_4 and transverse to S_b . Then the preimage $f^{-1}(S_b)$ will consist of $\partial(\mathcal{W} \times \mathbb{D})$, which is mapped to S_b with degree 1, and a collection of smooth hypersurfaces in the interior of $\mathcal{W} \times \mathbb{D}$.

If \mathcal{W} is not connected, it will have one component \mathcal{W}_∂ with boundary — the one containing the standard discs — and some closed components. The boundary component $\partial(\mathcal{W}_\partial \times \mathbb{D})$ of $\mathcal{W} \times \mathbb{D}$ is the one that maps with degree 1 onto S_b ; the other boundary components map with degree zero to S_b , since their image lies in the subset $S_b \setminus S^{2n-2} \times \{0\}$.

For $u \in \mathcal{W}$, the image point $u(1)$ lies in the $(2n - 1)$ -cell

$$\{(v + iw, \mathbf{s} + i\mathbf{t}, z) \in \mathbb{C} \times \mathbb{C}^{n-1} \times \mathbb{C}: v = 0, z = 1, w^2 + \frac{1}{2}(|\mathbf{s}|^2 + |\mathbf{t}|^2 + 1) \leq (R - \delta)^2\}.$$

It follows that the image $f(\mathcal{W} \times \{1\})$ is contained in a $(2n - 1)$ -dimensional cell inside S_b .

5.2. The mapping degree. Write M'_i , $i = 1, 2$, for the closures of the connected components of $M_R \setminus S_b$. We choose the numbering such that a collar neighbourhood of $\partial(\mathcal{W} \times \mathbb{D})$ is sent to a neighbourhood of $S_b = \partial M'_1$ in M'_1 . Set $V = \mathcal{W}_\partial \times \mathbb{D}$, where \mathcal{W}_∂ was defined in point (5) of the forgoing section, and continue to write f for $f|_V$. Set $V_i = f^{-1}(M'_i)$. Then

$$V = V_1 \cup_{f^{-1}(S_b)} V_2.$$

We orient S_b as the boundary of M'_1 , and V such that its boundary is mapped by f with degree 1 onto S_b . Then $f_i := f|_{V_i}: (V_i, \partial V_i) \rightarrow (M'_i, \partial M'_i)$ has a well-defined mapping degree, and these degrees can also be computed as the degree of the restriction to the boundary, $f_i|_{\partial V_i}: \partial V_i \rightarrow \partial M'_i$.

Lemma 8. *The mapping degrees of f_1 and f_2 satisfy*

$$\deg f_1 - \deg f_2 = 1.$$

Proof. We have $\partial M'_1 = -\partial M'_2$ and $\partial V_1 = -\partial V_2 \sqcup \partial V$, and ∂V is mapped with degree 1 onto $\partial M'_1$. \square

In particular, the degree of at least one of f_1, f_2 is non-zero. Together with the following lemma, this yields information about the fundamental group. This lemma is formulated for manifolds without boundary as an exercise in [5, p. 130], so for the reader's convenience we include the simple proof.

Lemma 9. *Let $g: (P, \partial P) \rightarrow (Q, \partial Q)$ be a map between compact, oriented m -dimensional manifolds. We assume that Q and ∂Q are connected, so that the mapping degree $\deg g$ is defined and equal to $\deg g|_{\partial P}$. Likewise, P is assumed to be connected, and we define the fundamental groups $\pi_1(Q), \pi_1(P)$ with respect to respective base points $q_0 \in Q$ and $p_0 \in g^{-1}(q_0)$.*

Then $g_(\pi_1(P)) \subset \pi_1(Q)$ is a subgroup whose index divides $\deg g$. In particular:*

- (i) *If $\deg g \neq 0$ and $g_*(\pi_1(P))$ is trivial, then $\pi_1(Q)$ is a finite group.*
- (ii) *If $\deg g = \pm 1$, then $g_*: \pi_1(P) \rightarrow \pi_1(Q)$ is onto.*

Proof. Let $\pi: (\tilde{Q}, \partial \tilde{Q}, \tilde{q}_0) \rightarrow (Q, \partial Q, q_0)$ be the unique covering with characteristic subgroup $\pi_*(\pi_1(\tilde{Q}))$ equal to $g_*(\pi_1(P))$. Then g lifts to a map $\tilde{g}: (P, \partial P, p_0) \rightarrow (\tilde{Q}, \partial \tilde{Q}, \tilde{q}_0)$.

The degrees of these maps satisfy $\deg g = \deg \pi \cdot \deg \tilde{g}$ — this is clear from the homological definition of the mapping degree — and $\deg \pi$ is the index of $\pi_*(\pi_1(\tilde{Q})) = g_*(\pi_1(P))$ in $\pi_1(Q)$. \square

We now want to apply this lemma to the maps f, f_1, f_2 . First we observe that one of the conditions in Lemma 9 (i) is satisfied.

Lemma 10. *For any choice of base points, the groups $f_*(\pi_1(V))$ and $(f_i)_*(\pi_1(V_i))$, $i = 1, 2$, are trivial.*

Proof. The space $V = \mathcal{W}_\partial \times \mathbb{D}$ retracts onto $\mathcal{W}_\partial \times \{1\}$, which is mapped by f to a cell inside S_b . This proves the statement for $f_*(\pi_1(V))$.

Given a based loop γ in V_i , the loop $f_i \circ \gamma$ is contractible in M_R . Since $\pi_1(M'_i)$ injects into $\pi_1(M_R) = \pi_1(M'_1) * \pi_1(M'_2)$, the loop $f_i \circ \gamma$ must represent the trivial element in $\pi_1(M'_i)$. \square

The manifolds V_1, V_2 are not, in general, connected. Still, we can speak of $\deg f_i$ as the sum of the degrees of the restriction of f_i to the components of V_i .

Proposition 11. *The maps $f_i: V_i \rightarrow M'_i$, $i = 1, 2$, have the following properties:*

- (i) *If $\deg f_i \neq 0$, then $\pi_1(M'_i)$ is finite.*
- (ii) *If $\deg f_i = \pm 1$, then $\pi_1(M'_i)$ is trivial.*
- (iii) *If $\deg f_1 \neq 0$, then one of $\pi_1(M'_i)$ is trivial.*
- (iv) *If both $\deg f_1$ and $\deg f_2$ are non-zero, then $\pi_1(M)$ is finite.*

Proof. Lemma 9 applies to each component of V_i . If $\deg f_i \neq 0$, then the degree on at least one of these components is non-zero, and claim (i) follows with Lemma 10.

In fact, $\pi_1(M'_i)$ must be a finite group whose order divides the degree of each component of f_i , and hence their sum $\deg f_i$. This implies (ii).

For (iii) we form a new space \hat{V} by gluing V to a copy of M'_2 along their respective boundaries with the help of the degree -1 map $f|_{\partial V}: \partial V \rightarrow \partial M'_2$, i.e.

$$\hat{V} := (V + M'_2)/x \sim f(x) \text{ for } x \in \partial V.$$

The map $f: V \rightarrow M$ and the inclusion map $M'_2 \rightarrow M$ induce a map $\hat{f}: \hat{V} \rightarrow M$.

For computing the fundamental group and the homology of \hat{V} , it is best to think of this space as the union of three open subsets: the interior of V , the interior of M'_2 , and the mapping cylinder of f with open collars attached to its ends ∂V and $\partial M'_2$. One then sees that $H_{2n+1}(\hat{V}) \cong \mathbb{Z}$ and

$$\pi_1(\hat{V}) = (\pi_1(V)/\pi_1(\partial V)) * \pi_1(M'_2).$$

In particular, \hat{f} has a well-defined mapping degree, and the proof of Lemma 9 still goes through, even though \hat{V} may not be a manifold.

By Lemma 10 we have $\hat{f}_*(\pi_1(\hat{V})) = \pi_1(M'_2)$, and by construction $\deg \hat{f} = \deg f_1$, which by assumption is non-zero. With Lemma 9 we conclude that $\pi_1(M'_2)$ is a subgroup of finite index in $\pi_1(M) = \pi_1(M'_1) * \pi_1(M'_2)$. This can only happen if $\pi_1(M'_1)$ is trivial, or if $\pi_1(M'_2)$ is trivial and $\pi_1(M'_1)$ is finite. (The finiteness of $\pi_1(M'_1)$ is already guaranteed by (i).)

Statement (iv) follows immediately from (iii) and (i). \square

5.3. The homology of the M'_i . We now use the maps f_i to derive homological information about the manifolds M'_i .

Lemma 12. *If $\deg f_i \neq 0$, then for $k \geq 1$ the homology groups $H_k(M'_i; \mathbb{Z})$ are finite and only contain p -torsion for primes p that divide $\deg f_i$.*

Proof. Let V_i^j be a component of V_i , and $f_i^j := f_i|_{V_i^j}$. For a field \mathbb{F} of characteristic zero or coprime to $\deg f_i^j$, the homomorphism $(f_i^j)_*: H_k(V_i^j; \mathbb{F}) \rightarrow H_k(M'_i; \mathbb{F})$ is surjective for all k , cf. [4, Lemma 13].

On the other hand, f_* is the zero homomorphism in degrees $k \geq 1$ by the argument used to prove Lemma 10. This implies that the composition of $(f_i)_*$ with the inclusion homomorphism $H_k(M_i; \mathbb{F}) \rightarrow H_k(M; \mathbb{F})$ is the zero homomorphism. Since the inclusion homomorphism is injective, $(f_i)_*$ must itself be trivial.

In combination, we conclude that $H_k(M'_i; \mathbb{F})$ is trivial for $k \geq 1$ and characteristic of \mathbb{F} equal to zero or coprime to any $\deg f_i^j$. \square

5.4. Proof of Theorem 2. By Lemma 8, at least one $\deg f_i$ is non-zero. If M is simply connected and has torsion-free homology, Lemma 12 implies that the corresponding M'_i is a simply connected homology ball with boundary diffeomorphic to S^{2n} , and hence a ball by the h -cobordism theorem, cf. [4].

If $\pi_1(M)$ is infinite, then by Proposition 11 (iv), one of f_1, f_2 has degree zero, and by Lemma 8 the other has degree ± 1 . From Proposition 11 (ii) and Lemma 12 we conclude that the corresponding M'_i is a simply connected homology ball, and hence a ball as before.

It remains to consider the case where $\pi_1(M)$ is finite but non-trivial. This implies that M'_1 , say, has finite fundamental group, and M'_2 is simply connected. Then the universal covering \widetilde{M} is given by the universal covering \widetilde{M}'_1 of M'_1 , with a copy of M'_2 glued to each of its $|\pi_1(M'_1)| > 1$ boundary components. By Proposition 11 we have $\deg f_1 \neq 1$, and hence $\deg f_2 \neq 0$. If $(\deg f_1, \deg f_2) = (0, -1)$, then M'_2 is a simply connected homology ball. So from now on we may assume that both $\deg f_1$ and $\deg f_2$ are non-zero.

Choose a regular value $m_0 \in S_b \subset M$ for both f and $f|_{\partial V}$ as base point of M , and a preimage $v_0 \in \partial V$ as base point of V . Choose a preimage $\widetilde{m}_0 \in \widetilde{M}$ of m_0 under the covering projection $\pi: \widetilde{M} \rightarrow M$ as the base point of \widetilde{M} . Write M'_2 for the copy of M'_2 in \widetilde{M} that contains \widetilde{m}_0 in its boundary.

By Lemma 10, the map $f: (V, v_0) \rightarrow (M, m_0)$ lifts to a map

$$\tilde{f}: (V, v_0) \longrightarrow (\widetilde{M}, \widetilde{m}_0).$$

Write

$$\tilde{f}_1: (V_1, v_0) \longrightarrow (\widetilde{M}'_1, \widetilde{m}_0)$$

for the restriction $\tilde{f}|_{V_1}$. Notice that if V_1 is disconnected, this does not coincide with the lift of the map f_1 . The boundary ∂V is mapped with degree 1 onto S_b , so the lifted map \tilde{f} sends ∂V with degree 1 onto the lifted copy of S_b containing the base point \tilde{m}_0 , i.e. the boundary of M_2^0 .

We have $\tilde{f}_1(\partial V_1) \subset \partial \widetilde{M}_1'$, so the mapping degree of \tilde{f}_1 is defined, and

$$\deg f_1 = |\pi_1(M_1')| \cdot \deg \tilde{f}_1.$$

Since f is transverse to S_b , any regular value for $f|_{\partial V_1}$ is also regular for f . So by Sard's theorem we can choose a regular value for \tilde{f}_1 on each boundary component, and $\deg \tilde{f}_1$ equals the number of preimages of any of these values.

Now consider the restriction of \tilde{f} to V_2 . This map sends each component of V_2 to one of the copies of M_2' in \widetilde{M} , with boundary mapping to boundary, so the mapping degree is defined. Choose a copy $M_2^1 \neq M_2^0$ of M_2' in \widetilde{M} , and write $\tilde{f}_2^0, \tilde{f}_2^1$ for the restriction of \tilde{f} to the preimage of M_2^0 or M_2^1 , respectively. A point in $\partial M_2^1 \subset \partial \widetilde{M}_1'$ has preimages only in the interior of V ; a regular value in ∂M_2^0 also has a single preimage (counted with signs) in ∂V . Since the total number of preimages of any regular value in \widetilde{M}_1' equals $\deg \tilde{f}_1$, we have

$$\deg \tilde{f}_2^1 = \deg \tilde{f}_1, \quad \deg \tilde{f}_2^0 = \deg \tilde{f}_1 - 1.$$

Now repeat the argument of Lemma 12 with the map

$$\pi \circ \tilde{f}_2^1: (\tilde{f}_2^1)^{-1}(M_2^1) \longrightarrow M_2'.$$

This shows that $H_k(M_2')$ can only contain p -torsion for primes p that divide

$$\deg(\pi \circ \tilde{f}_2^1) = \deg \tilde{f}_1,$$

and hence $\deg f_1$. But p must also divide $\deg f_2 = \deg f_1 - 1$. Once again, we deduce that M_2' is a simply connected homology ball.

This concludes the proof of Theorem 2.

Remark 13. Certain conclusions can be drawn in the simply connected case even in the presence of torsion. We illustrate this by an example. The 5-dimensional Brieskorn manifold $\Sigma(2, 3, 3, 3)$ is a simply connected spin manifold with $H_2 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$, see [11] or [8]; these data characterise the manifold up to diffeomorphism. Brieskorn manifolds carry natural contact structures, cf. [3, Section 7.1]. Let M_1, M_2 be two copies of this manifold, equipped with arbitrary contact structures ξ_1, ξ_2 , and let (M, ξ) be their contact connected sum. Then any contact form defining ξ must have a contractible periodic Reeb orbit, for otherwise we could speak of the evaluation maps f_1, f_2 , and the prime 2 would divide both $\deg f_1$ and $\deg f_2 = \deg f_1 - 1$.

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MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, WEYERTAL 86–90, 50931 KÖLN, GERMANY

E-mail address: `geiges@math.uni-koeln.de`

MATHEMATISCHES INSTITUT, WWU MÜNSTER, EINSTEINSTRASSE 62, 48149 MÜNSTER, GERMANY

E-mail address: `kai.zehmisch@uni-muenster.de`